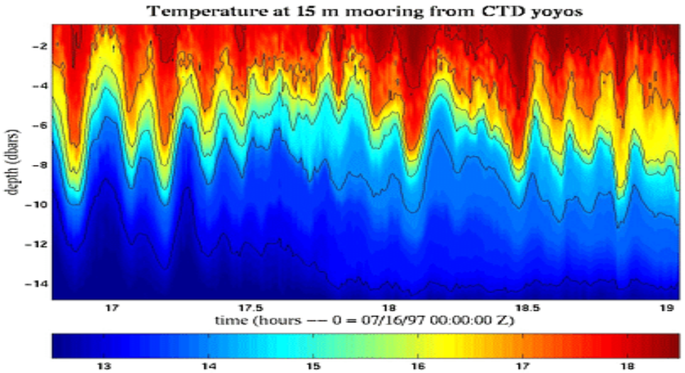
Capillary Waves (this class) and Internal Waves (next class)

Using similar wave equations as linear **surface gravity waves**

(Shallow Water Waves, SWW, and Deep-Water Waves, DWW)

to expand the solutions to more "exotic" waves



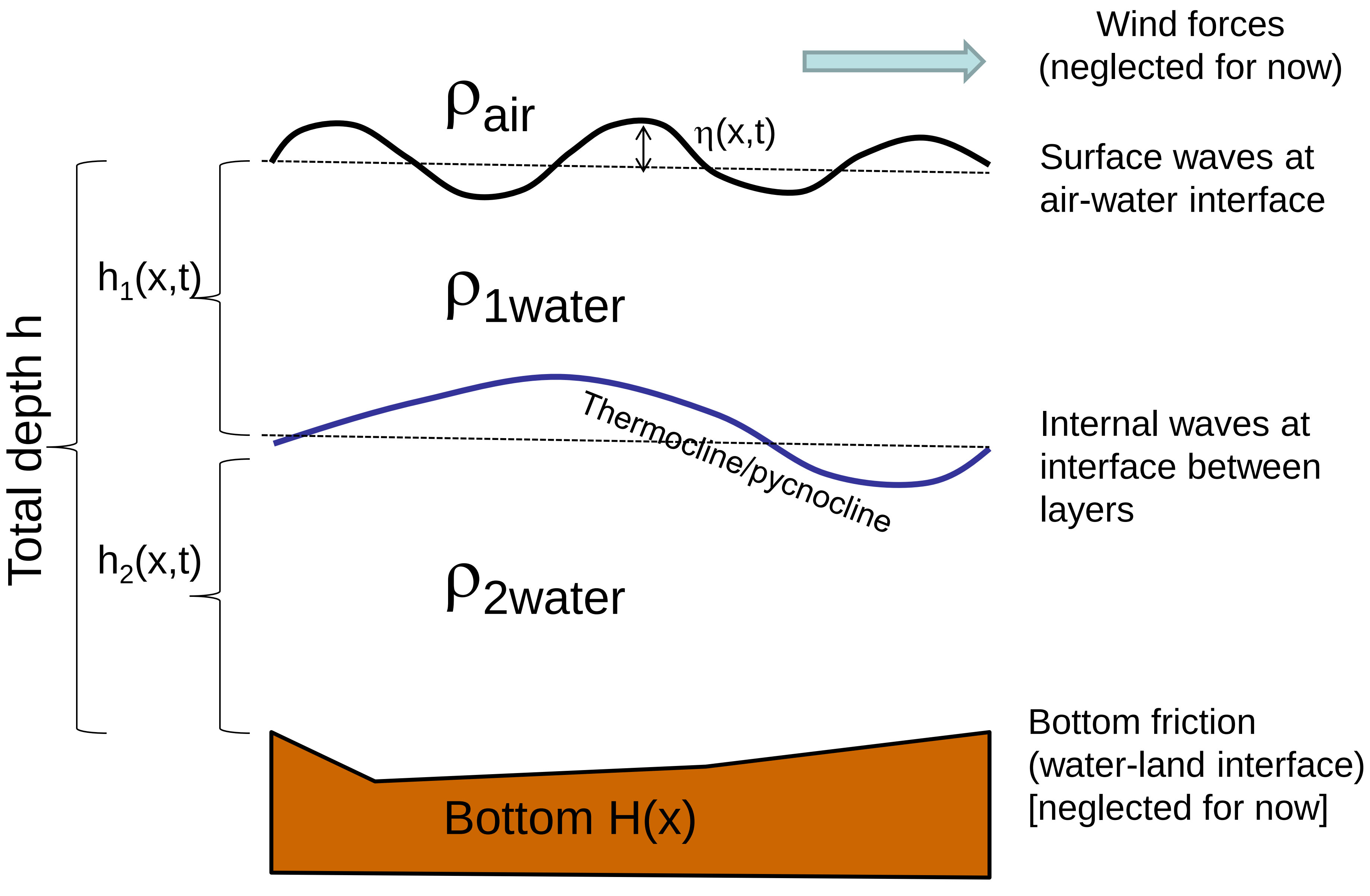
So far, we dealt with **surface gravity waves** while neglecting several important factors:

- Friction (bottom, surface wind, coast, etc.)
- Non-linearity (assumed "small waves")
- Earth rotation (Coriolis)
- Surface tension
- Stratification (assumed constant density)

Now let's look at the type of waves obtained when the last two conditions are relaxed…

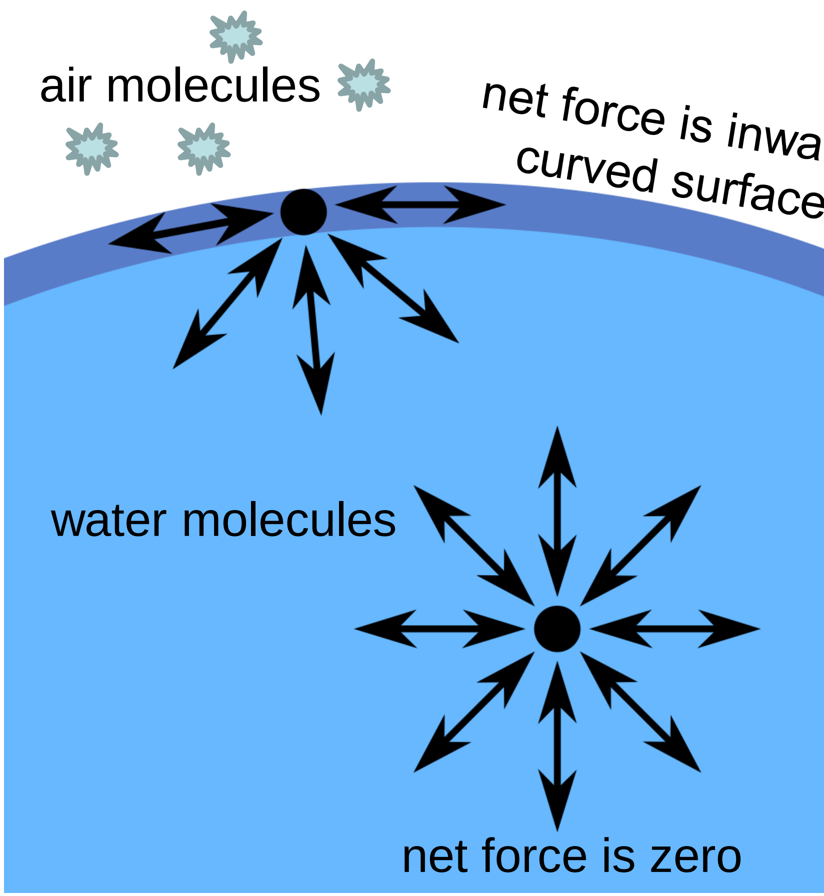
This will be an example how solutions of the same basic equations correspond to very different types of waves…

Forces acting on a 2-layer system



Surface tension is the force at the interface between layers of fluids (air-water; water with different densities; different liquids, etc):

- Force caused by cohesive attraction of similar molecules.
- It will resist pressure applied at the surface that changes the surface curvature



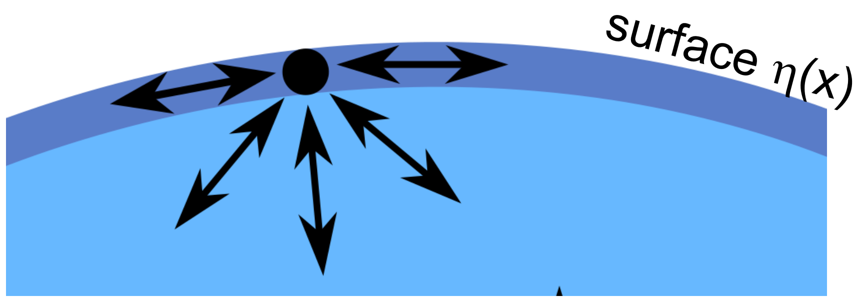
Surface tension allows Waterstrider to walk on water

Note that the net force of the surface tension depends on the surface **curvature**.

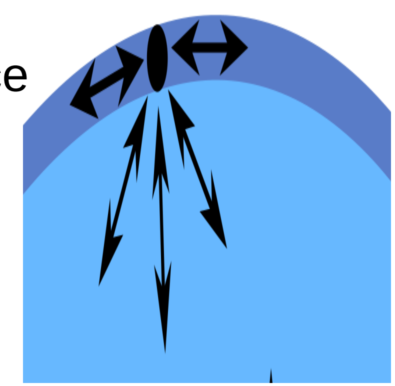
Curvature is proportional to 2nd derivative

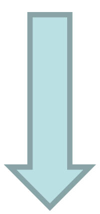
$$\frac{\partial^2 \eta}{\partial x^2}$$

small curvature

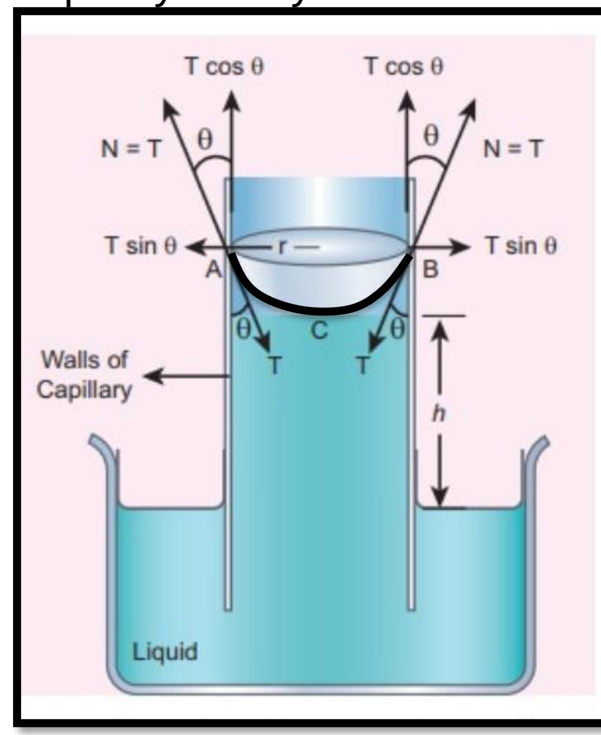


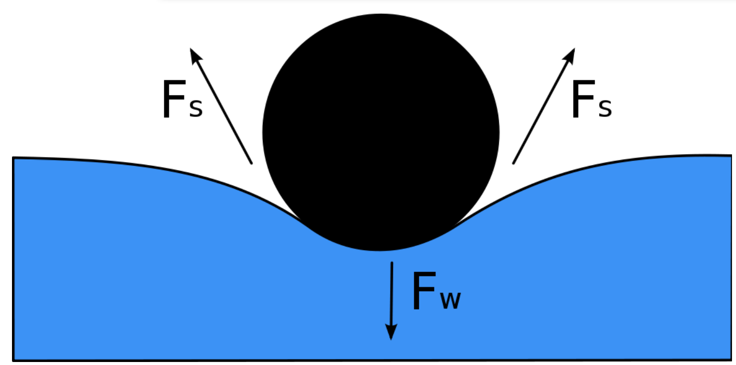
larger curvature:
larger net
downward force





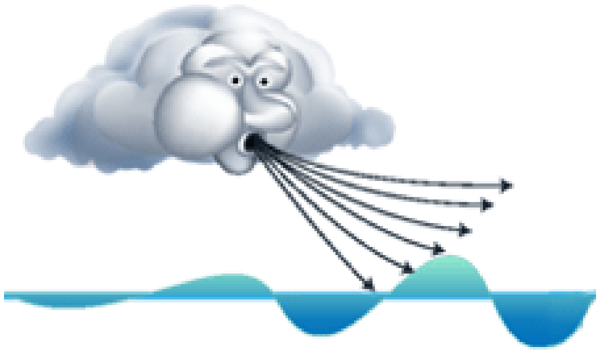
Capillary rise by surface tension



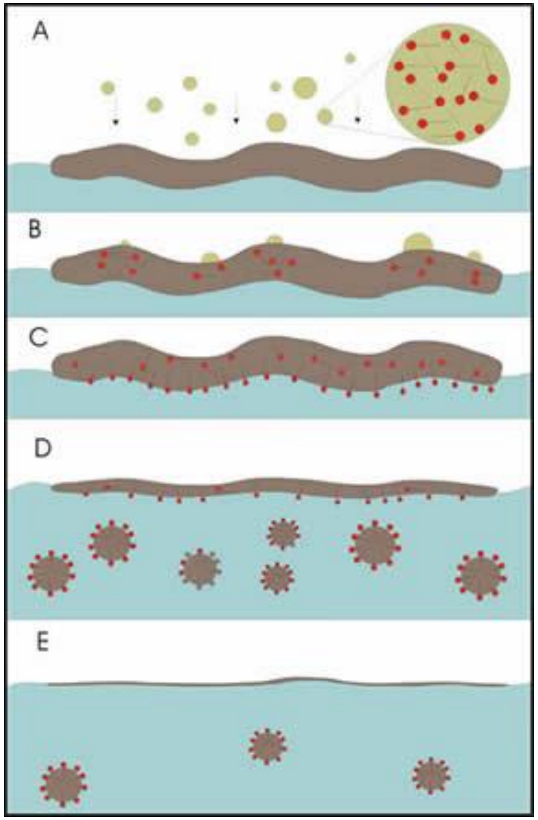


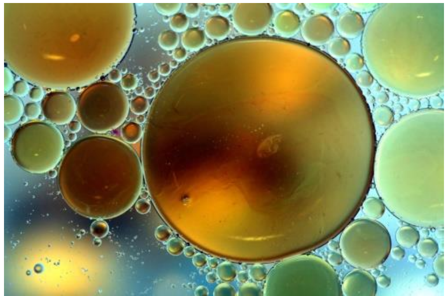
Importance of surface tension

- *Wind-driven ocean currents and surface waves*: transfer of energy from the wind to the ocean requires friction at the air-water interface



- *Oil spill dispersants* reduce the surface tension in the interface between the oil and the water, allowing easier mixing of the oil droplets





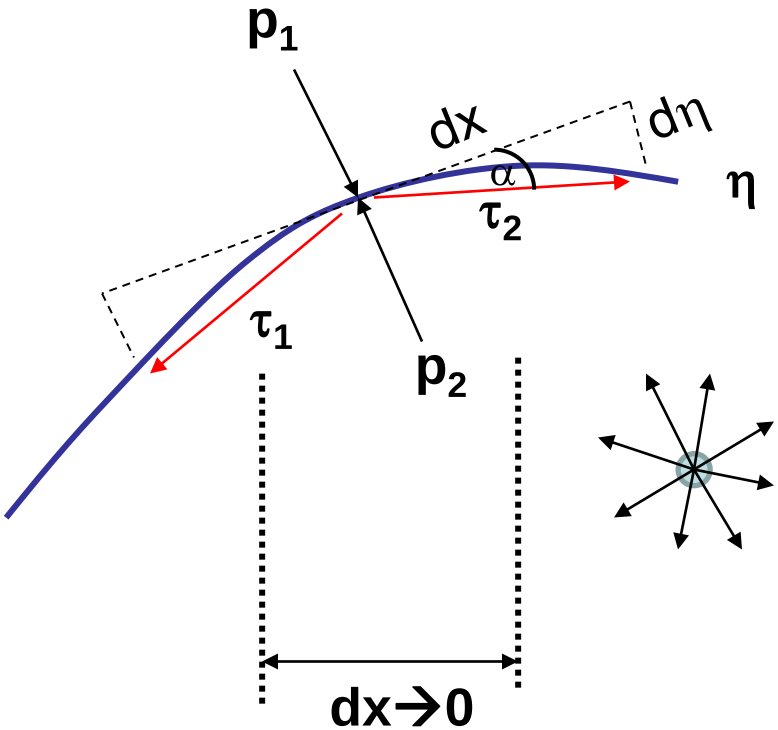
How to include Surface Tension forces?

The surface tension is tangent to the interface and balance with the pressure forces across the interface.

Under the assumption of small interface angle, the component of the force normal to the surface is:

$$\tau_2^{\,n} = -\tau_2 \sin\alpha \sim -\tau_2 \left.\frac{\partial \eta}{\partial x}\right|_2$$

And the pressure on area Δx is:



$$(p_1 - p_2)\Delta x \approx -\tau\left[\left.\frac{\partial \eta}{\partial x}\right|_2 - \left.\frac{\partial \eta}{\partial x}\right|_1\right]$$

or

at $z = \eta$, $$p_1 - p_2 \approx -\tau \frac{\partial^2 \eta}{\partial x^2}$$

τ = surface tension coefficient

Note: surface tension depends on the curvature of the interface (second derivative of the interface η), and molecular property of fluid (e.g., oil vs. water)

Back to the wave equations… reminder:

For an irrotational, non divergent flow the continuity and vorticity equations:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$

Can be represented using the stream function, $\Psi(x,z)$,

and

the velocity potential, $\phi(x,z)$

solve

$$u = \frac{\partial \Psi}{\partial z}, \quad w = -\frac{\partial \Psi}{\partial x}$$

substitude

solve

$$u = \frac{\partial \phi}{\partial x}, \quad w = \frac{\partial \phi}{\partial z}$$

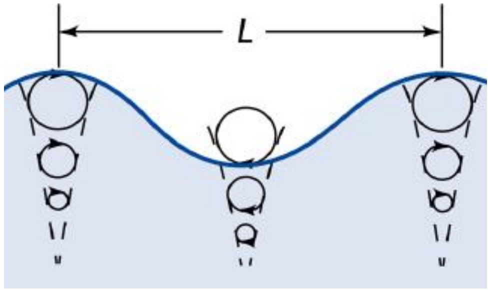
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial z^2} = 0$$

Applying boundary conditions to the **Laplace's equations** may provide solutions.

Previous example: 1-layer surface gravity waves

Next: more general solution for multi-layer cases



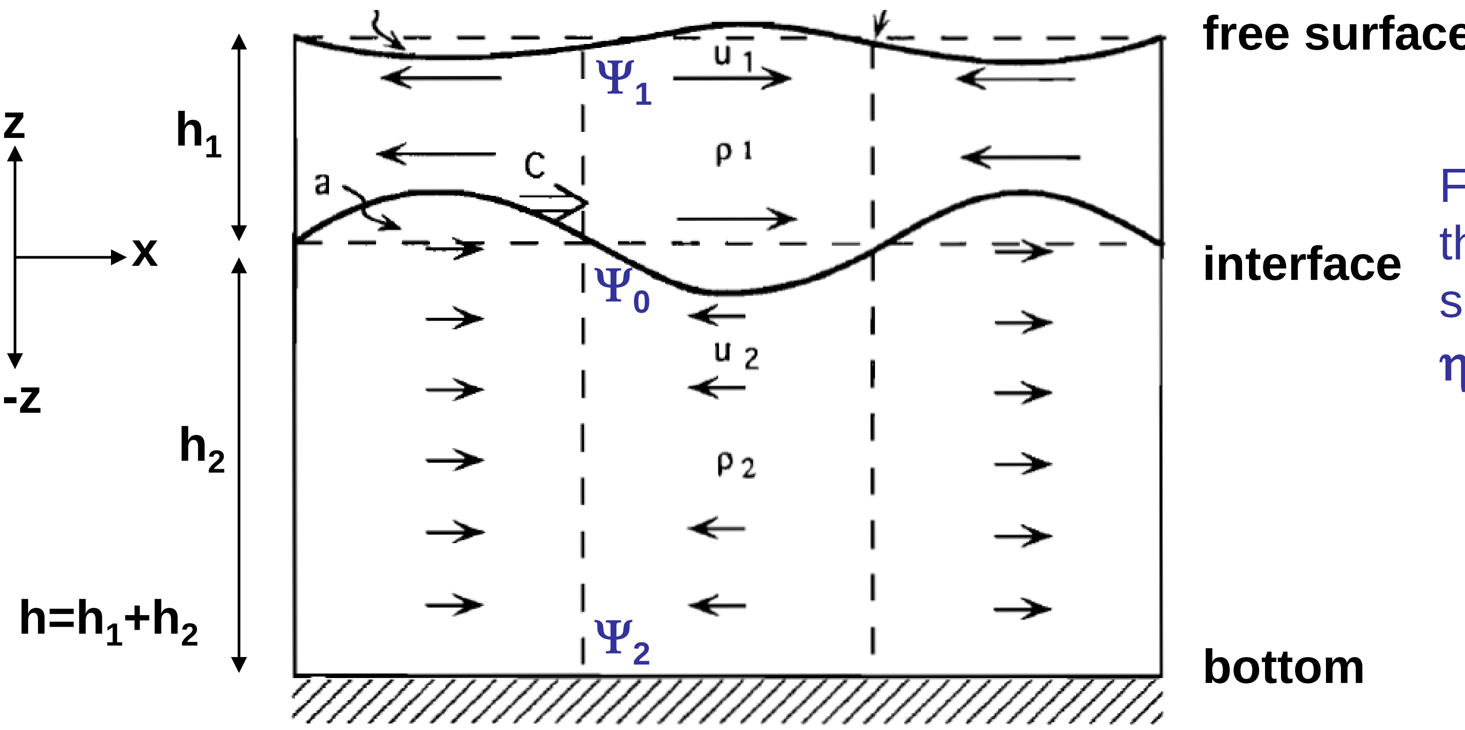
- The problem: two fluids of different densities separated by an interface
- We need to find solutions for the stream function $\Psi(x,z)$ and velocity potential $\phi(x,z)$, so the flow obeys the **Laplace Equations**:
- The **boundary conditions** are:

$$\nabla^2 \psi_1 = \nabla^2 \phi_1 = 0 \quad \text{(layer 1)}$$
$$\nabla^2 \psi_2 = \nabla^2 \phi_2 = 0 \quad \text{(layer 2)}$$

Surface & bottom: at $z=h_1$ $\Psi_1=C_1$; at $z=-h_2$ $\Psi_2=C_2$

(C_1 & C_2 are constants)

at the interface $z=\eta$ $\Psi_1=\Psi_2=0$, $g\eta=(\phi_1)_t-(p_1/\rho_1)=(\phi_2)_t-(p_2/\rho_2)$ (Bernoulli Eq.)



For linear waves the form of the solution is:
$\eta=a\cos(kx-\omega t)$

Assuming sinusoidal interface $\eta = a\cos(kx-\omega t)$ the solutions are:

$$\psi_1(x,z,t) = -ca\frac{\sinh\ k(z-h_1)}{\sinh\ kh_1}\cos(kx-\omega t)$$

$$\psi_2(x,z,t) = -ca\frac{\sinh\ k(z+h_2)}{\sinh\ kh_2}\cos(kx-\omega t)$$

[c = phase velocity of interface]

Satisfy boundary conditions:

- at the **surface** $z=h_1$ → stream func. $\psi_1=0$
- at the **bottom** $z=-h_2$ → stream func. $\psi_2=0$

- conditions at the interface:

(skip math…)

$$p_1 = (\rho_1 g + c^2\rho_1 k\coth kh_1)\eta$$

$$p_2 = (\rho_2 g + c^2\rho_2 k\coth kh_2)\eta$$

pressure relate to surface tension:

$$p_1 - p_2 \approx -\tau\frac{\partial^2\eta}{\partial x^2} = -\tau\frac{\partial^2}{\partial x^2}\left[a\cos(kx-\omega t)\right] =$$

$$= -\tau k^2\left[-a\cos(kx-\omega t)\right] = k^2\tau\eta$$

coth(x)=1/tanh(x)

Therefore using the above equations we get:

$$p_1 - p_2 = (\rho_1 g + c^2\rho_1 k\coth kh_1)\eta - (\rho_2 g + c^2\rho_2 k\coth kh_2)\eta = k^2\tau\eta$$

(dividing by $k\eta$ and rearranging terms …)

So now we can solve for the phase velocity of the interface:

$$c^2 = \frac{(\rho_2-\rho_1)\frac{g}{k} + k\tau}{\rho_1\coth kh_1 + \rho_2\coth kh_2}$$

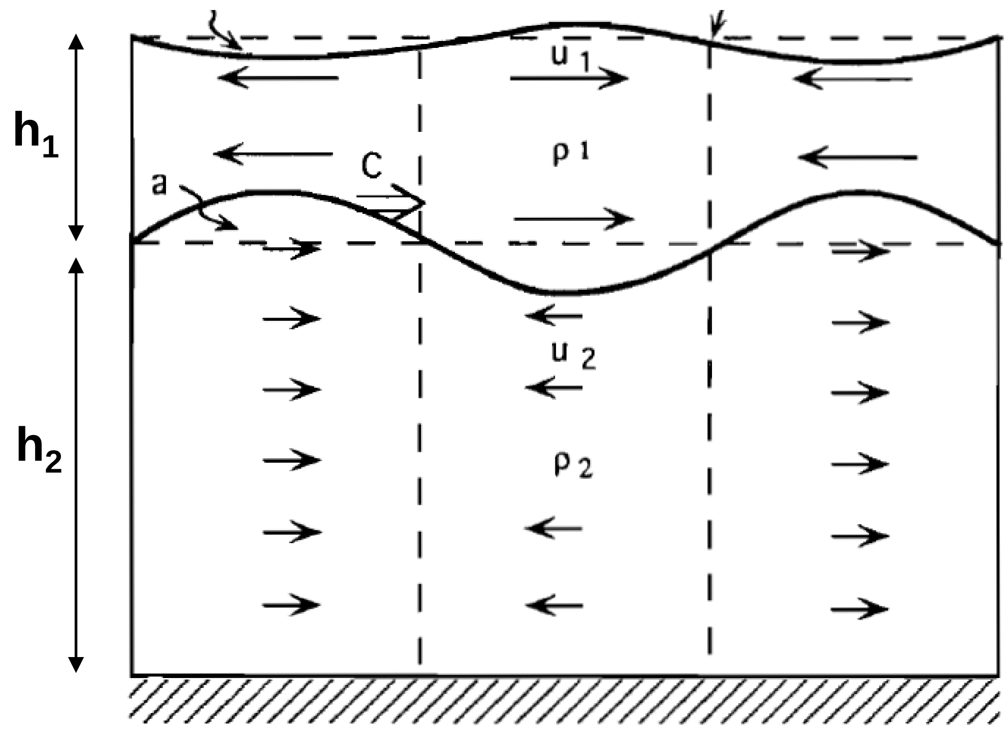
This is a general solution of the phase velocity of waves at the interface between two fluids, it depends on:

- Surface tension, τ
- Densities, ρ_1 & ρ_2
- Wavelength, $k = 2\pi/\lambda$
- Layer thicknesses, h_1 & h_2

$$c^2 = \frac{\overbrace{(\rho_2 - \rho_1)\frac{g}{k}}^{\text{gravity waves term}} + \overbrace{k\tau}^{\text{surface tension term}}}{\underbrace{\rho_1 \coth kh_1 + \rho_2 \coth kh_2}_{\text{shallow/deep layer term}}}$$

Under different conditions, different terms will be more important than others, describing different types of waves:

- Surface gravity waves
- Capillary waves
- Internal waves

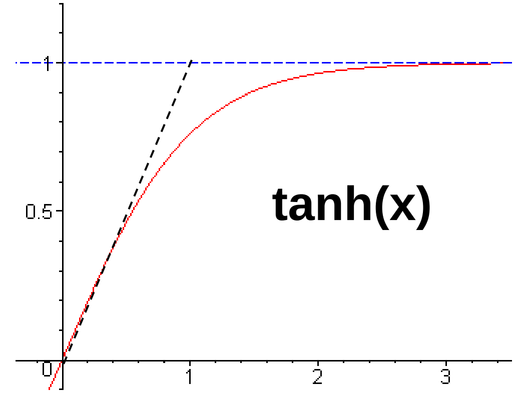


Surface gravity waves

$$c^2 = \frac{(\rho_2 - \cancel{\rho_1})\dfrac{g}{k} + \cancel{k\tau}}{\cancel{\rho_1 \coth kh_1} + \rho_2 \coth kh_2}$$

$$(kh = 2\pi h/\lambda)$$

coth(x)=1/tanh(x)



- First, for large enough waves $k\tau$ is small and we neglect surface tension
- Second, if the density of the upper layer is much smaller than the density of the lower layer (say $\rho_2=\rho_{water}$~1000 kg/m^3 and $\rho_1=\rho_{air}$~1 kg/m^3), then

$c^2=(g/k)\tanh(kh_2)$ upper layer (air) neglected, c independent of ρ

For **shallow-water** (or large waves) kh<<1, tanh(kh)~kh

$c^2=(gh_2)$ speed depends only on water depth (non dispersive waves)

For **deep-water** (or small waves) kh>>1, tanh(kh)~1 and

$c^2=(g/k)$ speed depends on wavelength (dispersive waves)

Therefore, we recovered the free surface solution we obtained before (it is just a special case of the more general equation!)

Capillary waves

$$c^2 = \frac{(\rho_2 - \rho_1)\dfrac{g}{k} + k\tau}{\rho_1 \coth kh_1 + \rho_2 \coth kh_2}$$

$$(kh = 2\pi h / \lambda)$$

- For small enough waves, the $k\tau$ term may be as large as the gravity term and we must consider the surface tension term.
- Also, if k is large, we can assume that kh>>1, i.e., deep-water waves.
- We still assume that the density of the upper layer is much smaller than the density of the lower layer $\rho_1 << \rho_2$ (consider only surface waves),

$$(\rho_2 - \rho_1)\frac{g}{k} \approx \rho \frac{g}{k} \qquad \rho_1 \coth kh_1 \approx 0 \qquad \rho_2 \coth kh_2 \approx \rho \times 1$$

Therefore, the phase speed of Capillary-gravity waves is:

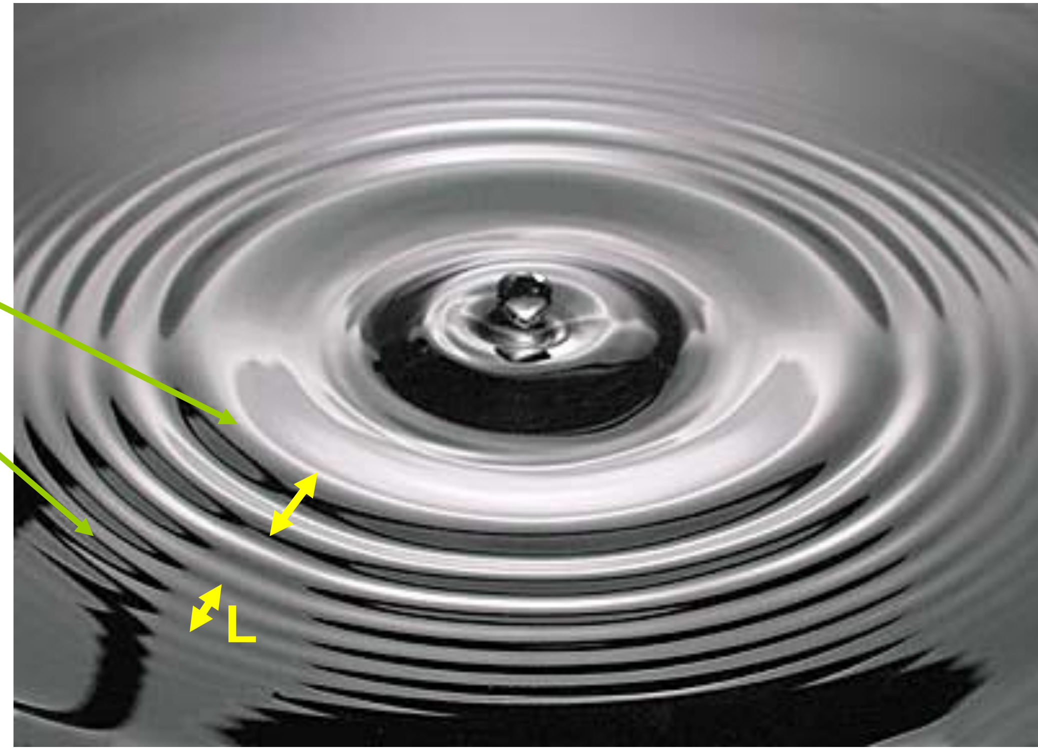
$$c^2 \approx \frac{g}{k} + \tau \frac{k}{\rho} = \left(\frac{g}{2\pi}\right)\lambda + \left(\frac{2\pi\tau}{\rho}\right)\frac{1}{\lambda}$$

Note that for large λ the gravity term dominates and longer waves move faster, but for small λ the surface tension term dominates and **shorter waves move faster!**

Capillary waves

Longer waves move slower

Shorter waves move faster



- Capillary waves are formed in the ocean under light wind conditions
- They create surface roughness that is important for the early stages of wind-wave growth

Capillary waves

To find the **minimum phase speed**

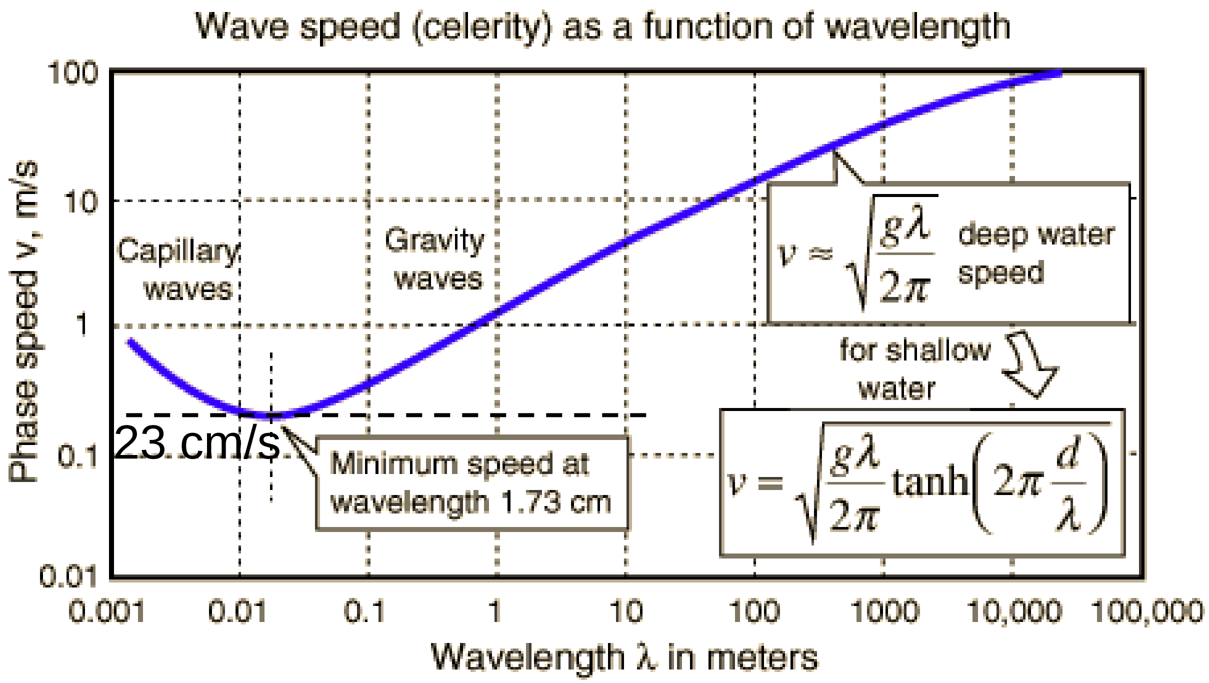
$$\frac{\partial c}{\partial k} = \frac{\partial}{\partial k}\left(\sqrt{\frac{g}{k} + \tau\frac{k}{\rho}}\right) = \frac{\overbrace{\left(-\frac{g}{k^2} + \frac{\tau}{\rho}\right)}^{=0}}{2\sqrt{\frac{g}{k} + \tau\frac{k}{\rho}}} = 0$$

and for typical values of
$g=9.8$ m/s^2, $\rho=1000$ kg/m^3 ,
$\tau=74$ dynes/cm = 0.074 N/m we get:

$$k_m = \sqrt{\frac{g\rho}{\tau}} = \sqrt{\frac{9.8 \times 1000}{0.074}} = 364\ \mathbf{m}^{-1}$$

$$\lambda_m = \frac{2\pi}{k_m} = 0.017\ m = 1.7\ \mathbf{cm}$$

$$c_m = \sqrt{\frac{9.8}{364} + 0.074\frac{364}{1000}} = 0.23\ \mathbf{m/s} = \mathbf{23\ cm/s}$$



$$C^2 = \overbrace{\frac{g}{k}}^{\textit{deep water waves}} + \overbrace{\frac{k}{\rho}\tau}^{\textit{surface tension correction}}$$

$$C^2 = \frac{g\lambda}{2\pi} + \frac{2\pi}{\rho\lambda}\tau$$

Wavelength of capillary waves with minimum phase speed:

$$\lambda_m = \frac{2\pi}{\sqrt{\frac{g\rho}{\tau}}}$$

How much do capillary waves affected by water temperatures and different fluids?

- Sea water at 0°C: τ=0.07564N/m, ρ=1028.11 kg/m^3 → λ=1.72 cm
- Sea water at 25°C: τ=0.07197N/m, ρ=1023.34 kg/m^3 → λ=1.68 cm

- Oil τ=0.020N/m, ρ=850 kg/m^3 → λ=0.97 cm
- Mercury τ=0.487N/m, ρ=13600 kg/m^3 → λ=1.2 cm

Fluids with larger surface tension are often also more dense so the effect on capillary waves is small

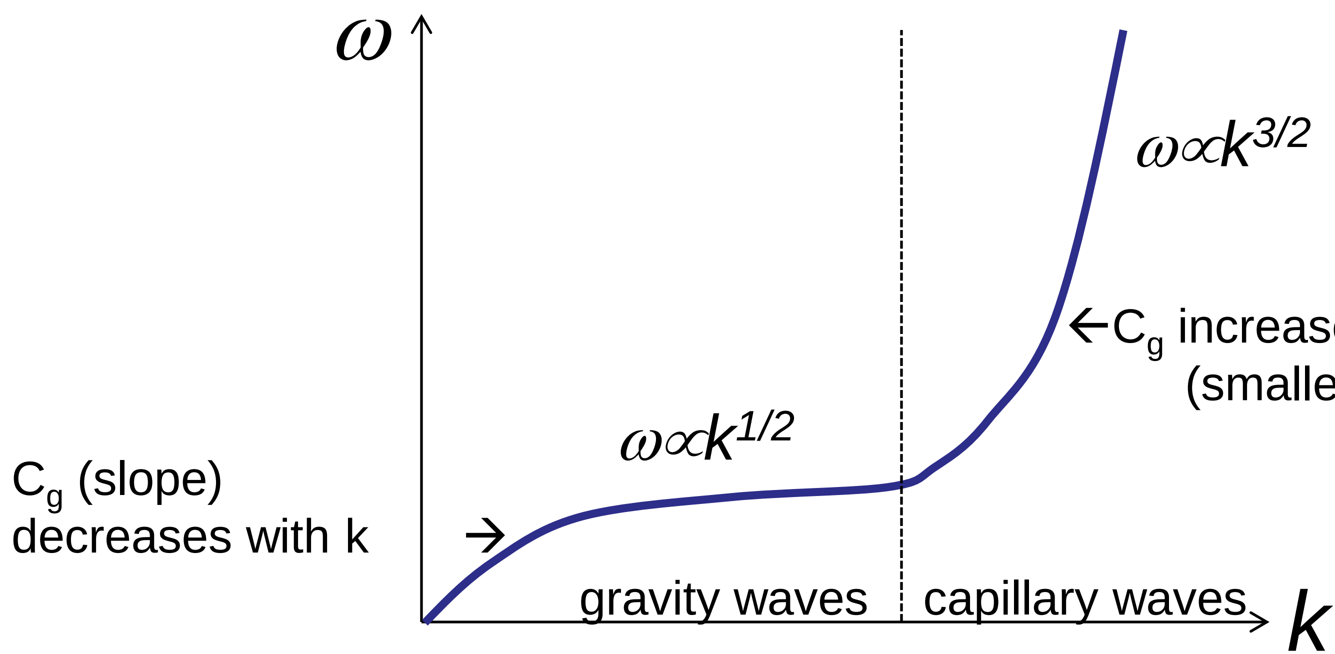
Note that salinity's effect on surface tension is also very small

Phase and group velocities of (deep-water) capillary-gravity waves

From the phase speed of capillary-gravity waves $c = \sqrt{\frac{g}{k} + \tau \frac{k}{\rho}}$

and the definition of phase and group velocities $c = \frac{\omega}{k}; \quad c_g = \frac{\partial \omega}{\partial k}$

We have the dispersion relation: $\omega = k\sqrt{\frac{g}{k} + \tau \frac{k}{\rho}} = \boxed{\sqrt{gk + \tau \frac{k^3}{\rho}}}$



Phase and group velocities

$$\omega = \sqrt{gk + \tau \frac{k^3}{\rho}}$$

From dispersion relation we can now calculate the phase and group velocities

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k} + \tau \frac{k}{\rho}} = \begin{cases} k << 1 \rightarrow \sqrt{\frac{g}{k}} & \text{(gravity waves)} \\ k >> 1 \rightarrow \sqrt{\tau \frac{k}{\rho}} & \text{(capillary waves)} \end{cases}$$

$$c_g = \frac{\partial \omega}{\partial k} = \frac{1}{2} \frac{g + 3\tau \frac{k^2}{\rho}}{\sqrt{gk + \tau \frac{k^3}{\rho}}} = \begin{cases} k << 1 \rightarrow \frac{1}{2}\sqrt{\frac{g}{k}} & \text{(gravity waves)} \\ k >> 1 \rightarrow \frac{3}{2}\sqrt{\tau \frac{k}{\rho}} & \text{(capillary waves)} \end{cases}$$

- **Gravity waves: group velocity is slower (1/2) than phase velocity**
- **Capillary waves: group velocity is faster (3/2) than phase velocity**

J. Fluid Mech. (1995), vol. 289, pp. 51–82

Capillary–gravity and capillary waves generated in a wind wave tank: observations and theories

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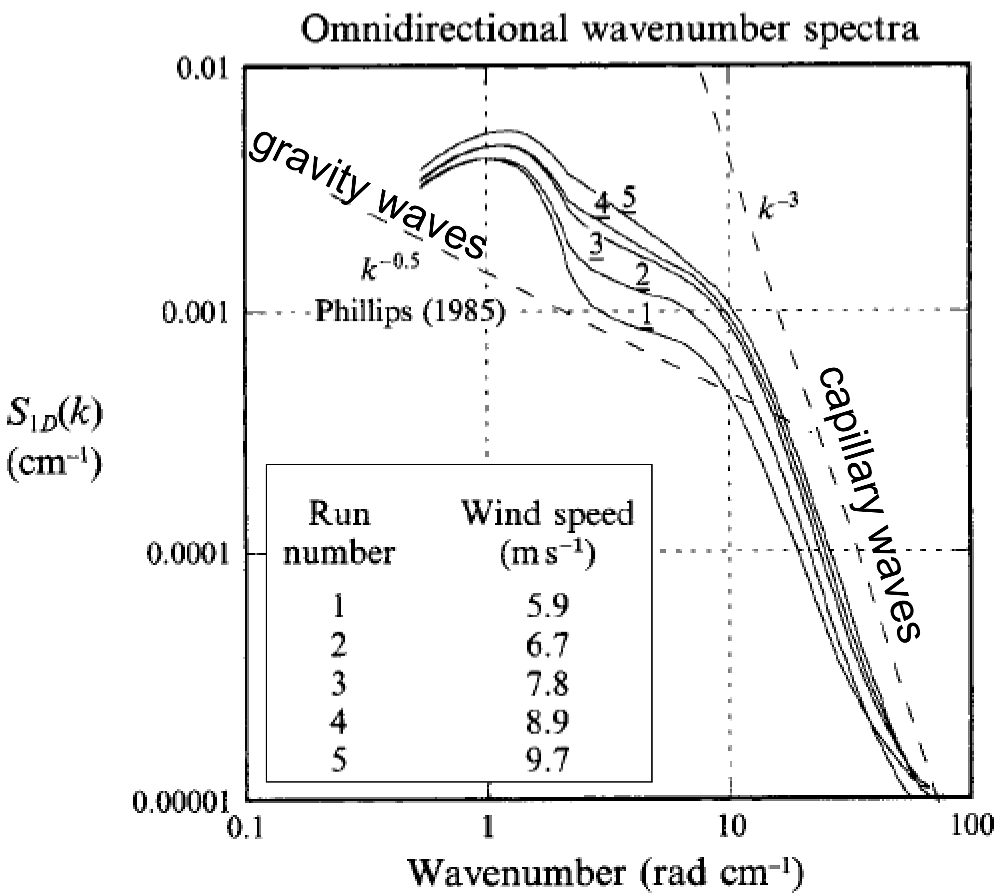
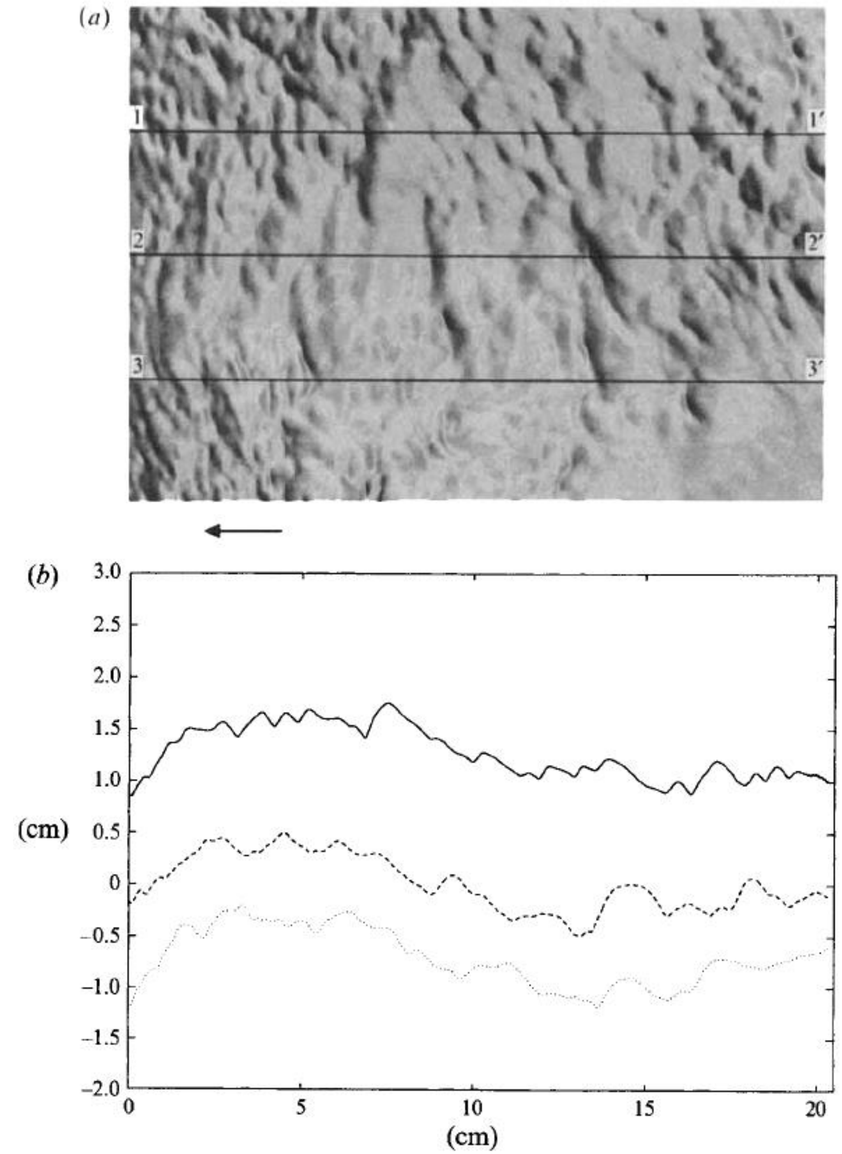
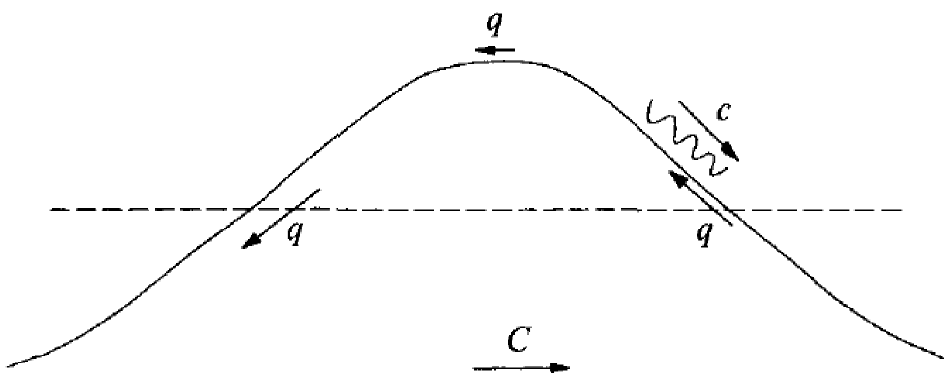
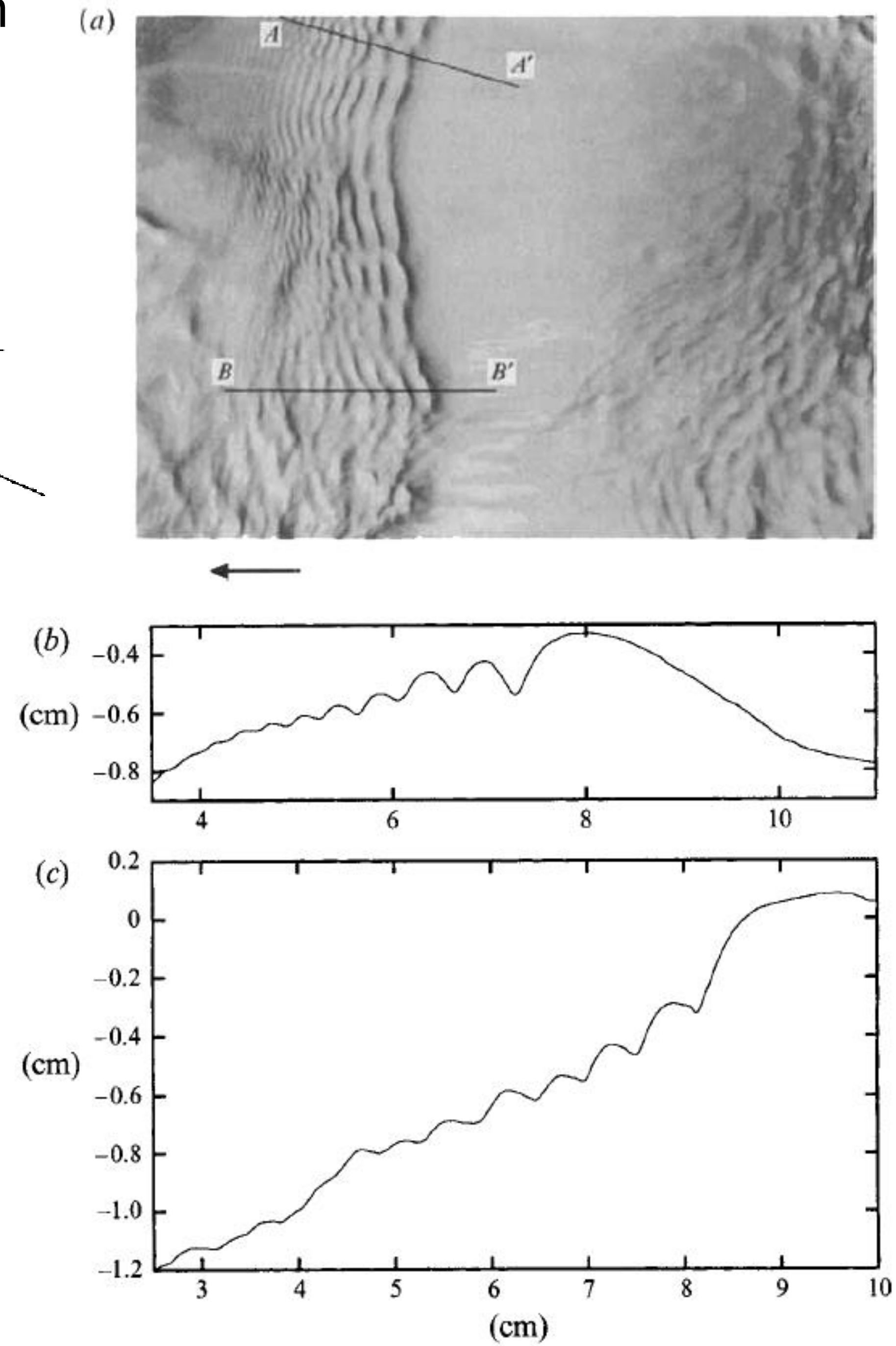


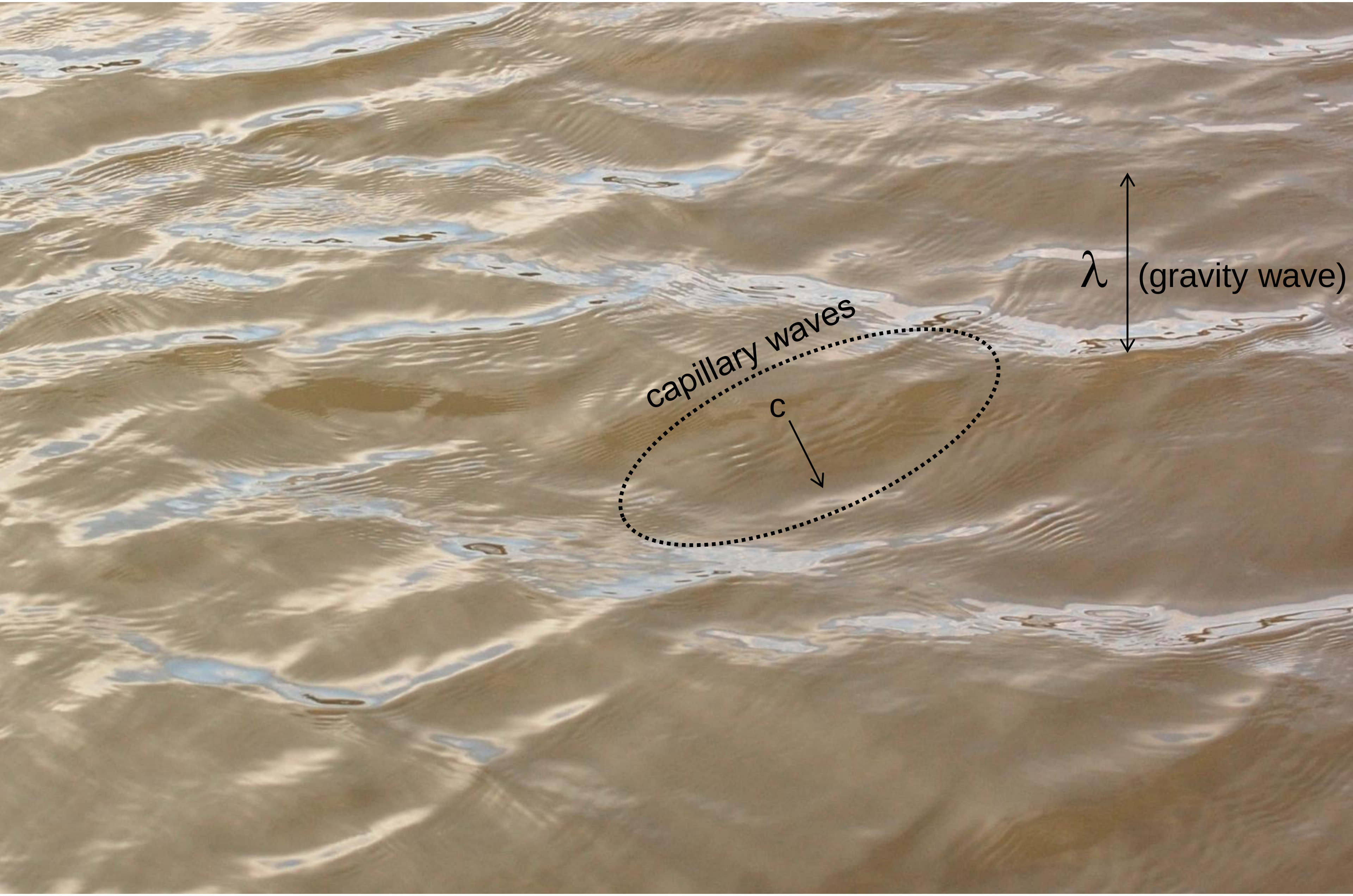


Short capillary waves can also ride on top of longer gravity waves





Wind-generated capillary waves riding on top of gravity waves




Water fountain-generated capillary waves

“duck-waves” (capillary)

Next Class: Internal Waves

$$c^2 = \frac{(\rho_2 - \rho_1)\frac{g}{k} + k\tau}{\rho_1 \coth kh_1 + \rho_2 \coth kh_2}$$

The same equation, but densities of the 2 layers are comparable $\rho_1 \sim \rho_2$

The solution depends on:

(a) $\Delta\rho=\rho_2-\rho_1 \rightarrow g'=g\Delta\rho/\rho$
(density of layers)

(b) h1,h2
(thickness of layers)

(c) $k=2\pi/\lambda$
(wavelength)